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On the surface area of the (n, k)-star graph

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ABSTRACT

We present an explicit formula for the *surface area* of the (n, k)-star graph, i.e., the *number of nodes at a certain distance from the identity node* in the graph, by identifying the unique cycle structures associated with the nodes in the graph, deriving a distance expression in terms of such structures between the identity node of the graph and any other node, and enumerating those cycle structures satisfying the distance restriction.

The above surface area derivation process can also be applied to some of the other node symmetric interconnection structures defined on the symmetric group, when the aforementioned distance expression is available.

1. Introduction

Given a graph $G(V, E)$ and a node $v \in V$, a question one may ask is *how many nodes are at distance d from v in G* for $d \in [1, D(G)]$, where $D(G)$ is the diameter of G. This quantity is known in the literature as the *surface area of G with radius d* [14]; or the *Whitney numbers of the second kind of the poset associated with G* [16].

One immediate application of a solution to the above problem is in computing various bounds for the problem of k-neighborhood broadcasting [13]. Such a solution can also be used to derive the *transmission of a (node symmetric) graph*, a notion recently suggested in [20] to achieve the generalized Moore bound, an important concept in extremal graph theory. As a result, this surface area problem has been studied for a variety of graphs including the star graph [16,17,14,21,22,5], the mesh structure [2], the k-ary n-cube [3], the rotator graph [11], the WK-recursive and swapped network [15]; and, very recently, for the arrangement graph, the alternating group graph, and the split-star graph [6–8]. It is clear that this problem has posed a challenge to both the applied and theoretical communities.

The surface area of any node u in a graph G with radius d is clearly equal to $|\{v|d_G(u, v) = d\}|$, where $d_G(u, v)$ stands for the distance between u and v in G. Since, for any nodes u, v in a node symmetric graph G, $d_G(u, v) = d_G(u_1, \phi(v))$, where u_1 is a node in such a graph G, and ϕ is an automorphism that maps u to u_1, the surface area for any node in such a graph G equals that for any other node in G. Hence, this quantity is especially well defined for the node symmetric graphs, which constitute the majority of the interconnection networks with mesh being one of the exceptions.

Let $\langle n\rangle$ denote $\{1, 2, \ldots, n\}$, $n \geq 2$, and let Γ_n be the symmetric group defined on $\langle n\rangle$, a *star graph with n dimensions*, S_n, is defined as a structure with its vertex set being Γ_n, and (u, v) is an edge in S_n iff for some $j \in [2, n]$, v can be obtained from u by applying a transposition $(1, j)$.

The star graph was proposed as an attractive alternative to the hypercube topology as an interconnection network, since it compares favorably with the latter structure in several aspects [1]. However, the requirement that the number of nodes in an n-star be $n!$ results in a large size gap between the n-star and the $(n + 1)$-star, which makes it impractical to use.

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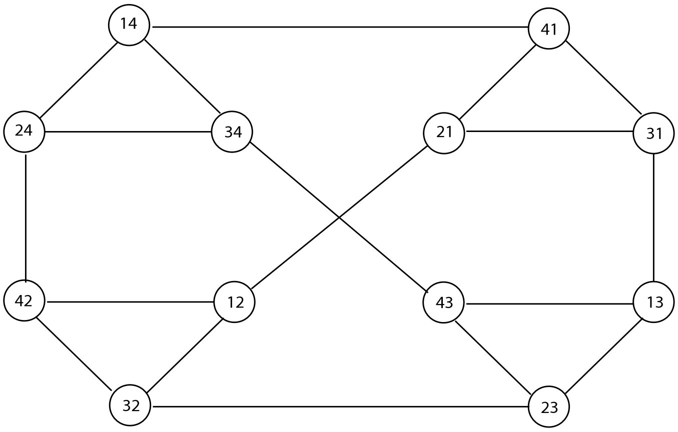

Fig. 1. A (4, 2)-star $S_{4,2}$.

To address this scalability issue, the (n, k)-star graph was put forward in [10], which does away with the above requirement on its size, while preserving many attractive properties of the star graph, including node symmetry [10, Theorem 3].

More specifically, in an (n, k)-*star graph*, $S_{n,k}(V, E)$, $k \in [1, n)$, $S_{n,k}$ for short, $V = \{u_1 \cdots u_k | \forall i \in [1, k], u_i \in \langle n \rangle$, *and* $u_i \neq u_j$ *if* $i \neq j\}$; *and, for any* $u, v \in V$, $(u, v) \in E$ *iff* v *can be obtained from* u *by either* (1) *applying a transposition* $(1, i)$ *to* u, $i \in [2, k]$ *(i-edge); or* (2) *for some* $x \in \langle n \rangle - \{u_i | i \in [1, k]\}$, *replacing* u_1 *with* x *in* u *(*1-edge*).*

For example, in a (4, 2)-star graph as shown in Fig. 1, (12, 21) is a 2-edge, while both (12, 32) and (12, 42) are 1-edges.

Since $S_{n,k}$ is node symmetric, its surface area equals that of $e(= 12 \cdots k)$, the identity node of $S_{n,k}$, which will be referred to as *the surface area of* $S_{n,k}$ in the rest of this paper. Hence, for $n \geq 2$, $k \in [1, n)$, and $d \in [1, D(S_{n,k})]$, let $B(n, k; d)$ stand for *the surface area of* $S_{n,k}$ *with radius d*, we have

$$B(n, k; d) = |\{v | v \in S_{n,k} \text{ and } d_{S_{n,k}}(e, v) = d\}|.$$

In this paper we will present an explicit formula (a sum of standard and basic operations, including factorials) for $B(n, k; d)$, the surface area of the (n, k)-star graph. To the best of our knowledge, this is among the first results of this nature reported in the literature.[1] As a crucial step of deriving such a formula, we will prove a distance formula for any two nodes in the (n, k)-star graphs via a minimum routing algorithm. As aforementioned, Chiang et al. was the first to study these graphs [10], where, among other things, they presented a routing algorithm, but did not prove its optimality. In this paper, we further expand their ideas to construct such an algorithm for the general case, and rigorously prove its optimality, using a result of Denes [12], a fundamental one in group theory.

The rest of this paper proceeds as follows: We characterize the unique *cycle structure* of any node in $S_{n,k}$ in Section 2; derive a distance expression between v, actually, the cycle structure of v, and e, via a minimum routing algorithm in Section 3; and finally come up with an explicit formula for the surface area of an (n, k)-star graph in Section 4. We conclude this paper with a discussion on the generality of our approach in Section 5.

2. The cycle structure of a node in the (n, k)-star graph

Let $v = v_1 v_2 \cdots v_k$ be a node of $S_{n,k}$. For all $i \in [1, k]$, we refer to v_i as *a symbol occurring in* v, and i its *position in* v. Moreover, following the terms adopted in [10], we refer to v_i as an *internal symbol,* if $v_i \in [1, k]$; and as an *external symbol* if $v_i \in (k, n]$. Similarly, we refer to a position i as an *internal (external) position,* respectively, depending on whether $i \in [1, k]$, or $i \in (k, n]$.

We start with the following definition, which specifies the relationship between an external symbol and its related internal symbols.

Definition 2.1. Let e_m be an external symbol occurring in an internal position of $v \in S_{n,k}$, $m \geq 1$, E_{e_m}, *the external cycle associated with* e_m *in* v*,* is the following cycle:

$$E_{e_m} = (e_m; e_0, \ldots, e_{m-1}),$$

where (1) for all $j \in [0, m-1]$, $e_j \in [1, k]$; (2) for $j \in [1, m]$, the position of e_j in v is e_{j-1}; (3) the position of e_0 in v is e_m.

For example, in $v = 2968134 \in S_{9,7}$, 8 is an external symbol, $v_4 = 8$, $v_7 = 4$, and 7 does not occur in v. Although 8 is also an external position, thus undefined in $S_{9,7}$, if we *define* $v_8 = 7$, as we did in the above definition, we have $E_8 = (8; 7, 4)$, where ';' indicates an external position.[2] Similarly, $E_9 = (9; 5, 1, 2)$.

[1] After we originally reported this result in [23], we have also derived an alternative but equivalent expression of the surface area of the (n, k)-star graph by following a generating function approach [6].

[2] This form of external cycles will be made use of in the later enumeration of such cycles.

We note that, in an external cycle E_{e_m}, the position of e_0 is defined to be e_m, an external position, i.e., $e_0 \notin v$. Thus, the transposition $(1, e_m)$, $e_m \in (k, n]$, corresponds to a 1-edge: switching v_1 with e_0. On the other hand, for $j \in [0, m-1]$, if $e_j \neq 1$, then, $(1, e_j)$ corresponds to an i-edge, $i \in [2, k]$: switching v_1 with v_{e_j}, since e_j is an internal position.

We call an external cycle $E_{e_m}(= (e_m; e_0, \ldots, e_{m-1}))$ a *primary external cycle,* if, for some $j \in [0, m-1]$, $e_j = 1$; and express such a cycle in the following equivalent *canonical form*[3]*:* $(1, e_{j+1}, \ldots, e_m; e_0, \ldots, e_{j-1}.)$ For example, the canonical form of the above E_9 is $(1, 2, 9; 5)$.

The definition of an *internal cycle* for $v \in S_{n,k}$ is given as usual [18, Section 4.1]:

$$C = (d_1, \ldots, d_l),$$

such that (1) for all $j \in [1, l]$, $d_j \in [1, k]$; (2) for $j \in [1, l-1]$, the position of d_{j+1} in v is d_j; and (3) the position of d_1 in v is d_l.

An internal cycle contains at least two positions; and, if for some $j \in [1, l]$, $d_j = 1$, we call C a *primary internal cycle;* and express C in its *canonical form:* $(1, d_{j+1}, \ldots, d_l, d_1, \ldots, d_{j-1})$. For example, in $v = 2968134$, there is exactly one internal cycle $(3, 6)$, which is not primary.

Since, for all j, d_j is an internal position, the transposition $(1, d_j)$, $d_j \in [2, k]$, is also permitted in $S_{n,k}$, a d_j-edge to be exact.

Therefore, for all the positions occurring in either an internal or an external cycle, the two kinds of transpositions as presented in the definition of $S_{n,k}$ are unified in the form of $(1, j)$, $j \in [2, n]$, all permitted in $S_{n,k}$. This fact is crucial for the later discussion of a (minimum) routing algorithm for $S_{n,k}$.

We now present an explicit process of representing any node $v(= v_1 v_2 \cdots v_k) \in S_{n,k}$, $v \neq e$, in terms of a collection of (internal/external) cycles by extending it first to a permutation $v' = v'_1 v'_2 \cdots v'_n$, in Γ_n as follows:

1. For $j \in [1, k]$, we set v'_j to v_j.
2. For $j \in (k, n]$, noticing that the k-permutation $v(= v_1 v_2 \cdots v_k) \in S_{n,k}$ is a function v from $\{1, 2, \ldots, k\}$ into $\{1, 2, \ldots, n\}$ with $v(i) = v_i$, $i \in \{1, 2, \ldots, k\}$, we consider two cases:
   - If $j \notin \{v_1, v_2, \ldots, v_k\}$, set $v'_j = j$.
   - Otherwise, $j = v_i$ for some $i \in [1, k]$. Letting $r = j$, repeatedly set r to $v^{-1}(r)$ until $r \notin \{v_1, v_2, \ldots, v_k\}$, and set v'_j to this last value of r. We note that all these symbols r, except the last one, occurs in an internal position in v.
3. We then represent v' in the traditional cyclic form, and put any primary cycle, in its canonical form, as the very first cycle.

For example, taking $v = 2968134 \in S_{9,7}$, we extend it to a permutation in Γ_9 as follows: besides the first 7 symbols as shown in v, for the external symbol 8, as it occurs in position 4, which occurs in position 7, but 7 does not occur in v, we set $v'_8 = 7$. Similarly, we set $v'_9 = 5$ to get $v' = 296813475$. It is immediate that its equivalent cyclic form is $(1, 2, 9; 5)(4, 8; 7)(3, 6)$, where the first two cycles are external and the last one is internal.

We summarize the above discussion into the following observation, which is similar to the factorization result for any permutation in Γ_n [19, Theorem 3.3].

Proposition 2.1. *Every node $v \in S_{n,k}$, $v \neq e$, can be factorized into the following product of disjoint cycles, each containing at least two symbols:*

$$v = E_{e_{m_1}} \cdots E_{e_{m_p}} C_1 \cdots C_r, p + r > 0,$$

where $E_{e_{m_i}}$, $i \in [1, p]$, are the external cycles and C_j, $j \in [1, r]$, the internal ones. This factorization is unique, except for the order in which the cycles are written.

Proof. When $v \neq e$, at least one transposition has to be applied to change v to e, thus there is at least one cycle containing at least two symbols in its equivalent cyclic form.

We now show that for an external symbol, its associated cycle in the extended permutation coincides with the external cycle as defined in Definition 2.1. Let e_m be an external symbol occurring in v,

- If e_m does not occur in v, the extension process will set $v'(e_m) = e_m$, i.e., a fixed point, thus not included in the factorization. On the other hand, no external cycle is defined for such an external symbol in Definition 2.1, since it does not occur in an internal position.
- Otherwise, the extension process generates a sequence of positions, $r_1, r_2, \ldots, r_k$, such that $v_{r_1} = e_m$, $v_{r_2} = r_1, \ldots, v_{r_k} = r_{k-1}$, but r_k does not occur in v, when the process sets $v_{e_m} = r_k$. Thus, for all $j \in [1, k-1]$, r_{j+1}, the position of r_j is internal. Moreover, r_1, the position of e_m, is also internal. It is thus clear that the resultant cycle coincides with the external cycle associated with e_m, as defined in Definition 2.1, i.e., $(e_m; r_k, \ldots, r_1)$.

The internal cycles in such a factorization are constructed as usual. The uniqueness now follows from the fact that all such cycles are disjoint since v is a function from $[1, k]$ into $[1, n]$. $\square$

[3] This canonical form of external cycles, as well as a forthcoming one for the internal cycles, will be made use of in the discussion of the minimum routing algorithm for the (n, k)-star graph.

As a direct consequence of the above result, we can now identify any node v of a given (n, k)-star graph with its unique cyclic factorization, which we will refer to as its *cycle structure*, denoted as $\mathcal{C}(v)$, in the rest of this paper.

It is clear that a cycle, $C = (c_1, c_2, \ldots, c_q), q \geq 2$, can be factorized into a product of $q - 1$ transpositions: $(c_1, c_2) \circ (c_1, c_3) \circ \cdots \circ (c_1, c_q)$, which puts every symbol occurring in C to its original position in e. For example, for $v' = 296813475$, one of its cycles is $(1, 2, 9, 5) = (1, 2)(1, 9)(1, 5)$, and

$$296813475 \stackrel{(1,2)}{\rightarrow} 926813475 \stackrel{(1,9)}{\rightarrow} 526813479 \stackrel{(1,5)}{\rightarrow} 126853479.$$

The following result [12, Lemma 1] shows that the above product is a shortest one of this nature.

Lemma 2.1. *A cycle of degree k, $k \geq 2$, cannot be represented by the product of less than $k - 1$ transpositions.*

We note that, when a cycle is primary, without loss of generality, $c_1 = 1$, any transposition in the above shortest product is in the form of $(1, j)$. When a cycle is not primary, such a shortest product where all the transpositions are in the specified $(1, j)$ form is not readily available. Nevertheless, we have the following result [22, Lemma 3.1].

Lemma 2.2. *A minimal product of transpositions, all in the form of $(1, j)$, for a cycle $C = (c_1, c_2, \ldots, c_q)$, $q \geq 2$, such that 1 does not occur in C, is: $(1, c_1) \circ (1, c_2) \circ \cdots \circ (1, c_q) \circ (1, c_1)$, with its length being $q + 1$.*

For example, for the cycle $(3, 6)$, a shortest product of transpositions, all containing a 1, is $(1, 3)(1, 6)(1, 3)$. Thus,

$$126853479 \stackrel{(1,3)}{\rightarrow} 621853479 \stackrel{(1,6)}{\rightarrow} 321856479 \stackrel{(1,3)}{\rightarrow} 123856479.$$

3. A minimum routing algorithm

We now discuss a minimum routing algorithm that changes any node $v(\neq e) \in S_{n,k}$ to e, based on its cycle structure, in terms of a sequence of transpositions in the form $(1, j), j \in [2, n]$. It is clear that such a routing algorithm has to change all the internal symbols occurring in both internal and external positions in v to their respective ones in e.

3.1. The exact bound of a minimum routing algorithm for the internal symbols

Let v be a node of $S_{n,k}$, $g_I(v)$ ($g_E(v)$) be the number of internal(external) cycles included in $\mathcal{C}(v)$, $b_I(v)$ ($b_E(v)$) be the number of symbols as contained in the respective g_I (g_E) cycles, and let $b(v)$ be $b_I(v) + b_E(v)$, the total number of symbols that $\mathcal{C}(v)$ contains. When the context is clear, we will use an abbreviated version of these notations, e.g., g_I, instead of $g_I(v)$.

We first consider an internal cycle $C = (d_1, d_2, \ldots, d_l)$, $d_j \in [1, k]$, as contained in $\mathcal{C}(v)$. If C is primary, without loss of generality, $d_1 = 1$, then by Lemma 2.1, a minimum routing algorithm, also referred to as a *transition sequence* henceforth, for C consists of the following sequence of $l - 1$ $S_{n,k}$ permitted transpositions:

$$(1, d_2) \circ (1, d_3) \circ \cdots \circ (1, d_l).$$

If C is not primary, by Lemma 2.2, a minimum routing algorithm for C consists of the following $l + 1$ $S_{n,k}$ permitted transpositions:

$$(1, d_1) \circ (1, d_2) \circ (1, d_3) \circ \cdots \circ (1, d_l) \circ (1, d_1).$$

Since all the internal cycles are disjoint, we have the following result.

Lemma 3.1. *Let $t_I(v, S_{(n,k)})$ be a shortest transition sequence, permitted in $S_{n,k}$, that changes the positions of all the (internal) symbols as contained in the internal cycles of $\mathcal{C}(v)$ to those in e, then*

$$|t_I(v, S_{n,k})| = \begin{cases} b_I + g_I - 2, & \text{if one of the internal cycles is primary} \\ b_I + g_I, & \text{otherwise.} \end{cases}$$

We note that the above distance formula is the same as suggested for the star graph [1], where all the symbols are trivially internal.

3.2. A lower bound for the external symbol routing algorithm

We start by establishing a lower bound for any routing algorithm for $S_{n,k}$ that changes the positions of all the internal symbols included in all the external cycles as contained in $\mathcal{C}(v)$ to their respective ones in e.

Lemma 3.2. *Let $v \in S_{n,k}$, such that $\mathcal{C}(v)$ contains at least one external cycle, and let $t_E(v, S_{n,k})$ be a shortest transition sequence, permitted in $S_{n,k}$, that changes the positions of all the internal symbols as contained in the external cycles in $\mathcal{C}(v)$ to those in e, then*

$$|t_E(v, S_{n,k})| \geq \begin{cases} b_E - 1, & \text{if one of the external cycles is primary} \\ b_E + 1, & \text{otherwise.} \end{cases}$$

Proof. Let $E_{e_m}(= (e_m; e_0, \ldots, e_{m-1}))$ be an external cycle in $\mathcal{C}(v)$, and let τ be a minimum product of transpositions that places all the internal symbols, $e_0, \ldots, e_{m-1}$, to their respective positions in e, but $|\tau| < |E_{e_m}| - 1$.

- if τ places e_m in position e_0, then τ is a product of transpositions that expresses the cycle E_{e_m}. By Lemma 2.1, $|\tau| \geq |E_{e_m}| - 1$, contradicting the above assumption;
- otherwise, because a product of one or more transpositions is a surjective function, there exists a sequence of k, $1 \leq k \leq n - m + 1$, positions $f_1, f_2, \ldots, f_k$, such that $\tau(f_1) = e_m, \tau(f_2) = f_1, \ldots, \tau(f_k) = f_{k-1}, \tau(e_0) = f_k$. In other words, τ is a product of transpositions corresponding to a cycle $(e_0, \ldots, e_m, f_1, \ldots, f_k) \in \Gamma_n$. Again, by Lemma 2.1, $|\tau| \geq |E_{e_m}| + k - 1 \geq |E_{e_m}|$, also contradicting the above assumption.

Hence, the length of any product of transpositions that puts all the internal symbols of E_{e_m} in their respective positions in e is at least $|E_{e_m}| - 1$.

Let E_{e_m} be a primary cycle, then, for some $j \in [0, m-1], e_j = 1$. One of the minimum products of $S_{n,k}$ permitted transpositions corresponding to E_{e_m} is $\tau_1 = (1, e_{j+1}) \circ \cdots \circ (1, e_{m-1}) \circ (1, e_m) \circ (1, e_0) \circ \cdots (1, e_{j-1})$, which puts all the symbols occurring in E_{e_m}, including all the internal ones, into their respective positions in e. Clearly, $|\tau_1| = m = |E_{e_m}| - 1$.

If E_{e_m} is not primary, an argument similar to that used to prove Lemma 2.2 shows that the length of a minimum product of $S_{n,k}$ permitted transpositions that puts all the internal symbols to their respective positions in e is $|E_{e_m}| + 1$.

Hence, if $\mathcal{C}(v)$ contains a primary external cycle, $E_{e_{m_1}}$, then

$$|t_E(v, S_{n,k})| \geq (|E_{e_{m_1}}| - 1) + (|E_{e_{m_2}}| + 1) + \cdots + (|E_{e_{m_p}}| + 1) = b_E + p - 2 \geq b_E - 1,$$

since by the assumption of this result, $p \geq 1$.

Similarly, when none of the external cycles in $\mathcal{C}(v)$ is primary,

$$|t_E(v, S_{n,k})| = b_E + p \geq b_E + 1.$$

This concludes the proof of this result. □

3.3. A minimum routing algorithm for the external symbols

We now construct a routing algorithm for $S_{n,k}$ that realizes the above lower bound as established in Section 3.2. Such a routing algorithm is sketched in [10], which is essentially to consider all the external cycles as a whole. We now fill in the details for such a strategy, discuss its correctness and demonstrate its minimality.

Let $E_{e^1_m} = (e^1_m; e^1_0, e^1_1, \ldots, e^1_{m-1})$ and $E_{e^2_r} = (e^2_r; e^2_0, e^2_1, \ldots, e^2_{r-1})$ be two external cycles.

- If one of them is primary, without loss of generality, assume that $E_{e^1_m}$ is primary, namely, for some $j \in [0, m-1], e^1_j = 1$. We then express $E_{e^1_m}$ in its canonical form: $E_{e^1_m} = (1, e^1_{j+1}, \ldots, e^1_{m-1}, e^1_m; e^1_0, \ldots, e^1_{j-1})$. Then the routing algorithm will be the following:

$$\begin{aligned} E_{e^1_m} E_{e^2_r} &= (1, e^1_{j+1}) \circ (1, e^1_{j+2}) \circ \cdots \circ (1, e^1_{m-1}) \circ (1, e^2_r) \circ (1, e^2_0) \\ &\quad \circ \cdots \circ (1, e^2_{r-1}) \circ (1, e^1_m) \circ (1, e^1_0) \circ \cdots \circ (1, e^1_{j-1}) \\ &= (1, e^1_{j+1}, \ldots, e^1_{m-1}, e^2_r; e^2_0, e^2_1, \ldots, e^2_{r-1}, e^1_m; e^1_0, \ldots, e^1_{j-1}). \end{aligned}$$

As we observed earlier in constructing the cycle structures for nodes in $S_{n,k}$, for all $z_1 \in [0, m-1] - \{j\}$, $z_2 \in [0, r-1]$, $e^1_{z_1}, e^2_{z_2} \in [2, k]$; and $e^1_m, e^2_r \in (k, n]$. Hence, all the steps $(1, e^1_{z_1})$ and $(1, e^2_{z_2})$ are the i-edges, $i \in [2, k]$; and both $(1, e^1_m)$ and $(1, e^2_r)$ are 1-edges. In other words, all these transpositional steps are permitted in $S_{n,k}$, and so is this routing algorithm.

Moreover, since the position of e^1_{j+1} is 1, $v_1 = e^1_{j+1}$, thus, e^1_{j+1} ends up at its right position e^1_{j+1} after the first transposition $(1, e^1_{j+1})$ is applied; while e^1_{j+2}, the original value in position e^1_{j+1}, is now in position 1, which will then be moved to its correct position e^1_{j+2} with the next transposition. This process continues until the transposition $(1, e^1_{m-1})$ is applied, which puts e^1_{m-1} in position e^1_{m-1}, while moving e^1_m, the original value in position e^1_{m-1}, into position 1. This segment puts $e^1_{j+1}, e^1_{j+2}, \ldots, e^1_{m-1}$ into their respective correct positions.

Now, instead of moving e^1_m, an external symbol, into its correct position e^1_m, which is not defined in $S_{n,k}$, we move it to e^2_r, the position of another external symbol, and move e^2_0, the original value of position e^2_r, into 1, which will be moved into position e^2_0 with the next transposition, i.e., $(1, e^2_0)$. This process continues until the transposition $(1, e^2_{r-1})$, which puts value e^2_{r-1} into its correct position e^2_{r-1}, while moving the external symbol e^2_r, the original symbol in position e^2_{r-1}, into position 1. This segment of transpositions thus puts $e^2_0, e^2_1, \ldots, e^2_{r-1}$ into their respective correct positions.

We finally move this external symbol e^2_r to position e^1_m, while e^1_0, the original value in position e^1_m, is moved to 1, and will be put into its correct position e^1_0 by the next transposition. Again, this process will be repeated until the last transposition $(1, e^1_{j-1})$, which puts e^1_{j-1} into position e^1_{j-1}, while putting the original symbol of e^1_{j-1}, i.e., $e^1_j(= 1)$, into position 1. This segment again puts rest of the internal symbols in E_m, i.e., $e^1_0, \ldots, e^1_{j-1}, e^1_j(= 1)$ into their respective desired positions in e.

Hence, all the internal symbols in $E^1_{e_m}$ and $E^2_{e_r}$ are placed into their correct positions by the above routing mechanism, while the two external symbols, e^1_m and e^2_r, switched their positions.

The above strategy can be directly generalized to the case when the cycle structure contains more than two external cycles, by breaking the primary external cycle into two pieces, then concatenating all the external cycles into one cycle, with the two parts of the broken primary external cycle serving as the prefix and the suffix of this combined cycle, respectively.

As we observed, at the end of the associated routing process, all the internal symbols are in their respective positions in e, while the external ones will be cyclically shifted to the right. Further analysis shows that, for this general case, when for each external cycle $E_{e_{m_i}}$ containing $m_i + 1$ external symbols, one being primary, it takes m_i transpositions, with a total of $b_E - g_E$ transpositions; and for each external cycle other than the primary one, it takes an additional transposition, $g_E - 1$ in total.

Hence, the total number of transpositions needed for this case is $b_E - 1$, realizing the lower bound as proved in Lemma 3.2. We can thus conclude that the above general routing algorithm is minimum for this case.

- For the other case, when none of the external cycles is primary, we have to somehow convert the involved cycles into a sequence of transpositions permitted in $S_{n,k}$. Let $E^1_{e_m} = (e^1_m; e^1_0, e^1_1, \ldots, e^1_{m-1})$ and $E^2_{e_q} = (e^2_q; e^2_0, e^2_1, \ldots, e^2_{q-1})$ be two external cycles, and none of them is primary. By Lemma 2.2, it turns out that the following slightly patched routing algorithm is minimum for this case with $m + q + 3$ transposition steps:

$$\begin{aligned} E^1_{e_m}E^2_{e_q} &= (1, e^1_m) \circ (1, e^1_0) \circ (1, e^1_1) \circ \cdots \circ (1, e^1_{m-1}) \circ (1, e^2_q) \\ &\quad \circ (1, e^2_0) \circ (1, e^2_1) \circ \cdots \circ (1, e^2_{q-1}) \circ (1, e^1_m). \end{aligned}$$

It can also be directly generalized, and it is clear that this strategy takes two extra steps as compared with the previous one, a total of $b_E + 1$ routing steps, again realizing the lower bound as established in Lemma 3.2, thus minimal.

We summarize the above discussion into the following result:

Corollary 3.1. *Let $t_E(v, S_{n,k})$ be a shortest transition sequence, permitted in $S_{n,k}$, that changes the positions of all the internal symbols as contained in the external cycles in $\mathcal{C}(v)$ to those in e, then*

$$|t_E(v, S_{n,k})| = \begin{cases} b_E - 1, & \text{if one of the external cycles is primary} \\ b_E + 1, & \text{otherwise.} \end{cases}$$

3.4. A distance expression for the (n, k)-star graph

Since all the cycles as contained in $\mathcal{C}(v)$ are disjoint, and a minimum $S_{n,k}$ permitted routing algorithm that changes the positions of all the internal symbols as contained in $\mathcal{C}(v)$ to those in e corresponds to a shortest path between v and e in $S_{n,k}$, based on Lemma 3.1 and Corollary 3.1, we have derived the following result.

Theorem 3.1. *The distance between e and v in $S_{n,k}$ can be expressed as follows:*

1. *If v does not contain any external cycle, then*

$$d_{S_{n,k}}(e, v) = \begin{cases} (a)\ b_I + g_I, & \text{if none of the internal cycles is primary,} \\ (b)\ b_I + g_I - 2. & \text{otherwise.} \end{cases} \tag{1}$$

2. *Otherwise,*

$$d_{S_{n,k}}(e, v) = \begin{cases} (a)\ b + g_I + 1 & \text{if none of the cycles is primary,} \\ (b)\ b + g_I - 1, & \text{otherwise.} \end{cases} \tag{2}$$

To summarize all the notions and results that we have discussed so far, Table 1 lists, for the graph $S_{4,2}$ as shown in Fig. 1, all the nodes, their extensions in Γ_4, their cycle structures, as well as their respective distances from e.

4. An explicit formula for the surface area of $S_{n,k}$

Let $B_1(n, k; d)$ refer to the number of nodes whose cycle structure contains no external cycles (Case 1 of Theorem 3.1), and $B_2(n, k; d)$ refer to that for those nodes whose cycle structure contains at least one external cycle (Case 2 of Theorem 3.1), we have

$$B(n, k; d) = B_1(n, k; d) + B_2(n, k; d).$$

Furthermore, let

$$B_2(n, k; d) = B_{21}(n, k; d) + B_{22}(n, k; d),$$

where $B_{21}(n, k; d)$ is the number of nodes whose cycle structure contains at least one external cycle and zero or more internal cycles, but none of them is primary; and $B_{22}(n, k; d)$ is the number of nodes whose cycle structure contains at least one external cycle and zero or more internal cycles with one of them being primary.

Table 1
An example for the distance formula.

v	v'	$\mathcal{C}(v)$	b_I	b_E	b	g_I	g_E	$d_{S_{n,k}}(e, v)$	Justification
21	21$\underline{34}$	(1, 2)	2	0	2	1	0	1	Eq. (1)(b)
32	32$\underline{14}$	(1, 3;)	0	2	2	0	1	1	Eq. (2)(b)
42	42$\underline{31}$	(1, 4;)	0	2	2	0	1	1	Eq. (2)(b)
31	31$\underline{24}$	(1, 3; 2)	0	3	3	0	1	2	Eq. (2)(b)
41	41$\underline{32}$	(1, 4; 2)	0	3	3	0	1	2	Eq. (2)(b)
23	23$\underline{14}$	(1, 2, 3;)	0	3	3	0	1	2	Eq. (2)(b)
24	24$\underline{31}$	(1, 2, 4;)	0	3	3	0	1	2	Eq. (2)(b)
13	13$\underline{24}$	(3; 2)	0	2	2	0	1	3	Eq. (2)(a)
14	14$\underline{32}$	(4; 2)	0	2	2	0	1	3	Eq. (2)(a)
43	43$\underline{21}$	(1, 4;) (3; 2)	0	4	4	0	2	3	Eq. (2)(b)
34	34$\underline{12}$	(1, 3;) (4; 2)	0	4	4	0	2	3	Eq. (2)(b)

When a cycle structure $\mathcal{C}(v)$ contains no external cycles, all the symbols occurring in v are taken from $[1, k]$. The only kind of transposition in a shortest transition sequence that changes v to e is in the form of $(1, j), j \in [2, k]$, i.e., the only transposition allowed in S_k, a star graph of dimension k. Therefore, these nodes $v \in S_{n,k}$, where $d_{S_{n,k}}(e, v) = d$, are exactly those nodes v in S_k such that $d_{S_{n,k}}(e, v) = d$. In other words,

$$B_1(n, k; d) = B_S(k; d),$$

where $B_S(k; d)$ refers to the surface area of the star graph S_k with radius d.

We have derived and then proved in [22] the correctness of the following result: for all $k \geq 2, d \in [1, D(S_k)]$, where $D(S_k)$ equals $\left\lfloor \frac{3(k-1)}{2} \right\rfloor$ as given in [1],

$$B_S(k; d) = \sum_{g_I=\max\{1,d-k+1\}}^{\lfloor \frac{d}{3} \rfloor} \binom{k-1}{d-g_I} \mathbf{d}(d-g_I, g_I) + \sum_{g_I=\max\{1,d-k+2\}}^{\lfloor \frac{d+2}{3} \rfloor} \binom{k-1}{d-g_I+1} \mathbf{d}(d-g_I+2, g_I). \tag{3}$$

In the above, $\mathbf{d}(m, l)$, *the number of ways of factorizing m symbols into l cycles,* $l \geq 2$, is discussed in [18, Section 4.4]. Based on Eqs. 4.18 and 4.19 [18]: for $m \geq 2l \geq 1$,

$$\mathbf{d}(m, l) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} s(m-j, l-j), \tag{4}$$

where $s(\_, \_)$ stands for the signless Stirling numbers of the first kind, and can be further represented as an explicit formula [14, Eqs. (5) and (6)].

We now come to the more interesting case when the cycle structure of a node, $v \in S_{n,k}$, does contain at least one external cycle:

$$\mathcal{C}(v) = (e^1_{m_1}; e^1_0, \ldots, e^1_{m_1-1}), \ldots, (e^p_{m_p}; e^p_0, \ldots, e^p_{m_p-1}), (d^1_1, \ldots, d^1_{l_1}), \ldots, (d^r_1, \ldots, d^r_{l_r}),$$

such that $p \geq 1, r \geq 0$, and for all $i \in [1, p], m_i \geq 1$, and for all $i \in [1, r], l_i \geq 2$.

We first construct and enumerate, in the following steps, those cycle structures for v, $d_{S_{n,k}}(e, v) = d$, that contain b symbols with $g_E(\geq 1)$ external cycles, $g_I(\geq 0)$ internal cycles, *and one of those cycles is primary*, when, by Case (b) of Eq. (2), $d = b + g_I - 1$.

1. Select g_E external symbols out of a total of $n-k$ such symbols in $\binom{n-k}{g_E}$ ways. To ensure that this binomial coefficient is not equal to 0, we require $n-k \geq g_E$, i.e.,

   $$1 \leq g_E \leq n-k. \tag{5}$$

   Since the order of the external cycles is of no significance, let those g_E external symbols be put down in an arbitrary but fixed manner.
2. Since one of the cycles is primary, 1 has to be chosen as one of the internal symbols, when selecting $b - g_E$ internal symbols out of a total of k such symbols. Hence, we choose $b - g_E - 1$ symbols out of $[2, k]$ in $\binom{k-1}{b-g_E-1}$ ways.
   Similar to the previous step, we require $k - 1 \geq b - g_E - 1$, i.e., $g_E \geq b - k$. Since $b = d - g_I + 1$,

   $$g_E \geq d - k + 1 - g_I. \tag{6}$$

   Since every internal cycle contains at least two symbols, $b_I \geq 2g_I$; and, since there is at least one external cycle, each of which contains at least two symbols, $b_E \geq 2$. Hence, $2g_I \leq b_I = b - b_E \leq b - 2 = d - g_I - 1$. In other words,

   $$g_I \leq \frac{d-1}{3}. \tag{7}$$

Combining Eqs. (6) and (7), we have that $g_E \geq d-k+1-g_I \geq d-k+1-\frac{d-1}{3} = \frac{2d-3k+4}{3}$, i.e., $g_E \geq \left\lceil \frac{2d-3k+4}{3} \right\rceil$. Combining this last lower bound for g_E with Eq. (5), we have the following bounds of g_E for this case:

$$\max\left\{1, \left\lceil \frac{2d-3k+4}{3} \right\rceil\right\} \leq g_E \leq n-k. \tag{8}$$

We can similarly derive the following bounds for g_I for this case:

$$\max\{0, d-n+1\} \leq g_I \leq \left\lfloor \frac{d-1}{3} \right\rfloor. \tag{9}$$

3. Once those $b-g_E$ internal symbols are chosen, we have to select b_E-g_E out of them, with or without 1, in $\binom{b-g_E}{b_E-g_E}$ ways, to add them into g_E external cycles, each containing one external symbol, arranged in an arbitrary but fixed order as chosen in Step 1.
We denote the number of partitioning those b_E-g_E symbols into g_E cycles, such that both the order of these cycles and that of the symbols inside each and every block matter, as $\mathbf{p}(b_E-g_E, g_E)$.
In general, *to insert $m(\geq 1)$ symbols into $b \in [1, m]$ blocks, each containing at least one symbol, and both the order of these blocks and those symbols within these blocks are important,* we first order those m symbols in $m!$ ways, then, for each such permutation, insert $b-1$ "slashes" to separate them while making sure that no two slashes are adjacent to each other. Hence,

$$\begin{aligned} &\mathbf{p}(1,1) = 1, \\ &\forall m \geq 1, b \in [1, m], \mathbf{p}(m, b) = m! \binom{m-1}{b-1}. \end{aligned} \tag{10}$$

We can also get the bounds for b_E, the number of symbols as contained in the g_E external cycles, as follows:

$$2g_E \leq b_E \leq d-3g_I+1, \tag{11}$$

4. We finally use the remaining $b-b_E$ internal symbols, with or without 1, to construct $g_I(\geq 0)$ internal cycles in $\mathbf{d}(b-b_E, g_I)(=\mathbf{d}(d-g_I-b_E+1, g_I))$ ways.

Therefore, we have achieved the following expression for the number of nodes whose cycle structure contains at least one external cycle(s) and 0 or more internal cycles with one of those cycles being primary.

$$\begin{aligned} B_{22}(n, k; d) &= \sum_{g_E, g_I, b_E} \binom{n-k}{g_E} \binom{k-1}{d-g_I-g_E} \binom{d-g_I-g_E+1}{b_E-g_E} \\ &\quad \mathbf{p}(b_E-g_E, g_E)\mathbf{d}(d-b_E-g_I+1, g_I), \end{aligned} \tag{12}$$

where the bounds of g_E, g_I and b_E are given in Eqs. (8), (9) and (11), respectively; and $\mathbf{d}(m, l)$ and $\mathbf{p}(m, n)$ as defined in Eqs. (4) and (10), respectively.

An explicit formula for the other case can be derived similarly, as follows:

$$\begin{aligned} B_{21}(n, k; d) &= \sum_{g_E, g_I, b_E} \binom{n-k}{g_E} \binom{k-1}{d-g_I-g_E-1} \binom{d-g_I-g_E-1}{b_E-g_E} \\ &\quad \mathbf{p}(b_E-g_E, g_E)\mathbf{d}(d-b_E-g_I-1, g_I). \end{aligned} \tag{13}$$

The bounds of g_E, g_I and b_E for $B_{21}(n, k; d)$ are given as follows:

$$\begin{aligned} &\max\left\{1, \left\lceil \frac{2d-3k+3}{3} \right\rceil\right\} \leq g_E \leq n-k, \ \max\{0, d-n\} \leq g_I \leq \left\lfloor \frac{d-3}{3} \right\rfloor, \\ &\text{and, } 2g_E \leq b_E \leq d-3g_I-1; \end{aligned}$$

We thus have the following main result of this paper, where $B_S(k; d)$, $B_{21}(n, k; d)$ and $B_{22}(n, k; d)$ are given in Eqs. (3), (13), (12), respectively; and the following expression for $D(S_{n,k})$ is derived in [10] (Incidentally, this diameter result can also be established by using Theorem 3.1.):

$$D(S_{n,k}) = \begin{cases} 2k-1, & \text{if } 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ k + \left\lfloor \frac{n-1}{2} \right\rfloor & \text{if } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq k < n. \end{cases}$$

Theorem 4.1. *The surface area of the (n, k)-star graph, $n \geq 2$, $k \in [1, n)$, is the following: for all $d \in [1, D(S_{n,k})]$,*

$$B(n, k; d) = B_S(k; d) + B_{21}(n, k; d) + B_{22}(n, k; d). \tag{14}$$

For example, $B(4, 2; 1) = 3$, $B(4, 2; 2) = 4$, and $B(4, 2; 3) = 4$, consistent with the results as shown in Table 1.

Table 2
Sample data for $B(8, k; d)$.

k	d									
	1	2	3	4	5	6	7	8	9	10
1	7	0	0	0	0	0	0	0	0	0
2	7	12	36	0	0	0	0	0	0	0
3	7	22	81	145	80	0	0	0	0	0
4	7	30	129	377	644	444	48	0	0	0
5	7	36	174	638	1634	2517	1,545	168	0	0
6	7	40	210	870	2710	5849	7,233	2,960	280	0
7	7	42	231	1015	3430	8379	13,083	10,408	3409	315

It is straightforward to come up with a computer program to calculate the surface area of $S_{n,k}$ based on Eq. (14). The data so obtained agree with what we found through a breadth-first search. Some sample data for $B(8, k; d)$, $k \in [1, 7]$, $d \in [1, D(8, k)]$, are collected in Table 2.

We note that none of these sequences is included in the On-line Encyclopedia of Integer Sequences [24], except the row corresponding to $B(8, 7; d)$, $d \in [1, 10]$. The latter sequence coincides with the surface area for S_8, since in general $S_{n,n-1}$ is isomorphic to S_n [10, Lemma 4].

5. Conclusion

We characterized the cycle structures of nodes in (n, k)-star graphs and used this structure to obtain an explicit formula to compute the surface area centered at the identity node in the graph for any radius. Such a formula will help in establishing various bounds in data communication on those structures, and the techniques developed in deriving such formulas should be found useful elsewhere in studying similar problems.

In general, a cycle structure of any node $v \in \Gamma_n$ corresponds to a shortest transition sequence that places all the symbols that are not fixed in v to their respective positions in the identity node $e \in \Gamma_n$; while the structural specificity of a node symmetric graph G, defined on Γ_n, often places certain restriction on the nature of the transpositions permitted in G, which *might* lead to $d_G(\mathcal{C}(v), e)$, a G dependent *distance formula in terms of the cycle structure of* v. The (n, k)-star graph that we have studied in this paper provides an example in this regard.

We hope that this paper has made it clear that, if we can somehow derive such a distance expression for such a graph G, we can readily follow the approach that we discussed in this paper to derive an explicit formula for the surface area of G. Indeed, by following this approach, we have derived explicit formulas for the respective surface area of the star graph [22], arrangement graph [7], alternating group graph [8], split-star graph [8] and alternating group network [9].

On the other hand, since no such distance expression for either the bubblesort graph or the pancake graph is known, this aforementioned approach cannot be readily used to derive their respective surface area, although it is clear that the surface area for a bubblesort graph defined on Γ_n with radius d is $\mathcal{I}(n, d)$, i.e., the number of permutations on $\langle n \rangle$ with d inversions. An "explicit formula" for $\mathcal{I}(n, d)$ is given in [4, Eq. 2.5], involving the *pentagonal numbers.*

Another matter of interest is that, once such a distance expression is found, besides following the direct counting approach, as discussed in this paper, to derive explicit formulas for the surface area of such a structure, we can also apply a generating function based approach to derive an alternative but equivalent result [5,6].

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References

[1] S.B. Akers, Krishnamurthy, A group theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (4) (1989) 555–566.
[2] H. Sarbazi-Azad, On some combinatorial properties of meshes, in: Proc. International Symp. on Parallel Architecture, Algorithms and Networks, ISPAN'04, IEEE Comp. Society, 2004, pp. 117–122.
[3] H. Sarbazi-Azad, M. Ould-Khaoua, L.M. Mackenzie, S.G. Akl, On the combinatorial properties of k-ary n-cubes, J. Interconnect. Netw. 5 (1) (2004) 79–91.
[4] M. Bóna, Combinatorics of Permutations, Chapman & Hall/CRC, Boca Raton, FL, 2004.
[5] E. Cheng, K. Qiu, Z. Shen, A short note on the surface area of star graphs, Parallel Process. Lett. 19 (1) (2009) 19–22.
[6] E. Cheng, K. Qiu, Z. Shen, A generating function approach to the surface area of some interconnection networks, Tech. Rep. 2008-03, Dept. of Mathematics and Statistics, Oakland Univ., Rochester, MI, USA, 2008.
[7] E. Cheng, K. Qiu, Z. Shen, On deriving explicit formulas of the surface areas for the arrangement graphs and some of the related graphs, Manuscript, 2008.
[8] E. Cheng, K. Qiu, Z. Shen, On the surface area of the alternating group graphs and the split star graphs, Tech. Rep. 2008-09, Dept. of Mathematics and Statistics, Oakland Univ., Rochester, MI, USA, 2008.

[9] E. Cheng, K. Qiu, Z. Shen, On the surface area of the alternating group networks, Manuscript, 2008.
[10] W. Chiang, R. Chen, The (n, k)-star graph: A generalized star graph, Inform. Process. Lett. 56 (1995) 259–264.
[11] P.F. Corbett, Rotator graphs: An efficient topology for point-to-point multiprocessor networks, IEEE Trans. Parallel Distrib. Syst. 3 (5) (1992) 622–626.
[12] J. Denés, The representation of permutation as the product of a minimal number of transpositions and its connection with the theory of graphs, Magy. Tud. Akad. Mat. Kutint. 4 (1959) 63–71.
[13] G. Fertin, A. Raspaud, k-neighbourhood broadcasting, in: 8th International Colloquium on Structural Information and Communication Complexity, SIROCCO'01, 2001, pp. 133–146.
[14] N. Imani, H. Sarbazi-Azad, S.G. Akl, Some topological properties of star graphs: The surface area and volume, Discrete Math. (2008) doi:10.1016/j.disc.2008.08.007.
[15] N. Imani, H. Sarbazi-Azad1, A.Y. Zomaya, Some properties of WK-recursive and swapped networks, in: Proc. Fifth International Symposium on Parallel and Distributed Processing and Applications, ISPA'07, in: LNCS, vol. 4742, 2007, pp. 856–867.
[16] F. Portier, T. Vaughan, Whitney numbers of the second kind for the star poset, European J. Combin. 11 (1990) 277–288.
[17] K. Qiu, S.G. Akl, On some properties of the star graph, VLSI Design 2 (4) (1995) 389–396.
[18] J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York, 1980.
[19] J. Rotman, The Theory of Groups, second ed., Allyn and Bacon, Inc., Boston, MA, 1973.
[20] M. Sampels, Vertex-symmetric generalized Moore graphs, Discrete Appl. Math. 138 (2004) 195–202.
[21] Z. Shen, K. Qiu, On the Whitney numbers of the second kind for the star poset, European J. Combinatorics 29 (7) (2008) 1585–1586.
[22] Z. Shen, K. Qiu, An explicit formula of the surface area for the star graph and a proof of its correctness, in: Congressus Numerantium (Proc. Thirty-Ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing, CGTC39, Boca Raton, FL, March 3–7, 2008), 192 (December 2008), pp. 115–127.
[23] Z. Shen, K. Qiu, E. Cheng, On the surface area of the (n, k)-star graph, in: Proc. of the 2nd Annual International Conference on Combinatorial Optimization and Applications, COCOA'08, in: LNCS, vol. 5165, Springer, St. Johns, Newfoundland, Canada, 2008, pp. 393–404.
[24] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences; software available at http://www.research.att.com/~njas/sequences/.